Fourier Analysis

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Review

Consider the Time-dependent heat equation on the real line

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = f(x). \end{cases}$$

Define

$$\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad x \in \mathbb{R},\ t > 0$$

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space.

Then

$$u(x,t) = f * \mathcal{H}_t(x), \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+$$

is a solution of the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial^2 u}{\partial x^2} \quad \text{on } \mathbb{R} \times \mathbb{R}_+ \\ \lim\limits_{t \to 0} u(x,t) = f(x). \end{cases}$$

However, it is not the unique solution. For instance,

$$\widetilde{u} = f * \mathcal{H}_t(x) + \frac{x\,\mathcal{H}_t(x)}{t}$$ is another solution.

Def. Given a function $u(x,t)$ on $\mathbb{R}\times\mathbb{R}_+$, we say that $u(\cdot,t)$ belongs to $S(\mathbb{R})$ uniformly in t if $u(\cdot,t)\in S(\mathbb{R})$ for all $t>0$, and moreover, for each $T>0$, and each $k,\ell\geq 0$.

$$\sup_{\substack{x\in\mathbb{R}\\ 0<t<T}} \left| x^k \cdot \frac{\partial^\ell u(x,t)}{\partial x^\ell} \right| < \infty .$$

Check: $\frac{x}{t}\cdot \mathcal{H}_t(x)$ does not belongs to $S(\mathbb{R})$ uniformly in t.

Because
$$\sup_{\substack{x\in\mathbb{R}\\ 0<t<T}} \left|\frac{x}{t}\,\mathcal{H}_t(x)\right| \geq \sup_{0<t<T} \left|\frac{\sqrt{t}}{t}\,\mathcal{H}_t(\sqrt{t})\right|$$

$$= \sup_{0<t<T} \frac{1}{\sqrt{4\pi}\, t}\cdot e^{-\frac{1}{4}}$$

$$= +\infty .$$

Thm (Uniqueness).

Let $u \in C^{\infty}(\mathbb{R} \times \mathbb{R}_+) \cap C(\mathbb{R} \times [0,\infty))$.
Suppose u satisfies the following properties.

① $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ on $\mathbb{R} \times \mathbb{R}_+$;

② $u(x,0) = 0$ for all $x \in \mathbb{R}$;

③ $u(\cdot, t) \in S(\mathbb{R})$ uniformly in t.

Then $u(x,t) = 0$ on $\mathbb{R} \times \mathbb{R}_+$.

Pf. (Energy method)

Define for $t \geq 0$,

$$E(t) = \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx$$

In particular, $E(0) = 0$, $E(t) \geq 0$ for $t \geq 0$.

Observe that

$$\frac{d}{dt}E(t) = \frac{d}{dt}\int_{-\infty}^{\infty} |u(x,t)|^2 dx$$

$$\overset{\text{By (DCT)}}{=} \int_{-\infty}^{\infty} \frac{d}{dt}|u(x,t)|^2 dx$$

Here we need to use $u \in C^\infty(\mathbb{R}\times\mathbb{R}_+)$ and $u(\cdot,t) \in \mathcal{S}(\mathbb{R})$ unif. in t.

$$= \int_{-\infty}^{\infty} \frac{d}{dt}\left(u(x,t)\,\overline{u(x,t)}\right) dx$$

$$= \int_{-\infty}^{\infty} \left(\partial_t u(x,t)\right)\cdot \overline{u(x,t)} + u(x,t)\,\overline{\partial_t u(x,t)}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}\cdot \overline{u} + u \cdot \frac{\partial^2 \overline{u}}{\partial x^2}\, dx$$

$$\overset{\text{Integration by Parts}}{=} \frac{\partial u}{\partial x}\overline{u}\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial u}{\partial x}\cdot\frac{\partial \overline{u}}{\partial x}\, dx$$

$$+ u\cdot\frac{\partial \overline{u}}{\partial x}\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial u}{\partial x}\cdot\frac{\partial \overline{u}}{\partial x}\, dx$$

$$= -2\int_{-\infty}^{\infty} \frac{\partial u}{\partial x} \cdot \frac{\partial \bar{u}}{\partial x}\, dx$$

$$= -2\int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x} \right|^2 dx$$

$$\leq 0.$$

Hence $E(t)$ is a non-increasing function on $(0, \infty)$.

But $E(0) = 0$, so $E(t) \leq 0$ for $t > 0$.

However, by definition $E(t) \geq 0$.

As a consequence,

$$E(t) \equiv 0 \text{ for all } t \geq 0.$$

That is

$$\int_{-\infty}^{\infty} |u(x,t)|^2 dx \equiv 0.$$

$$\Rightarrow \quad u(x,t) \equiv 0. \qquad \boxed{/\!/}$$

Prop. Let $f \in S(\mathbb{R})$ and let

$$u(x,t) = f * \mathcal{H}_t(x).$$

Then $u(\cdot,t) \in S(\mathbb{R})$ uniformly in t in the sense that

$$\sup_{\substack{x\in\mathbb{R} \\ 0<t<T}} \left| x^k \frac{\partial^\ell u(x,t)}{\partial x^\ell} \right| < \infty \qquad (**)$$

for any $T>0$ and $k, \ell \geq 0$.

Pf. Without loss of generality, we only prove $(**)$ in the case when $k=1$, $\ell=1$.

Let $T>0$.

$$\frac{\partial u(x,t)}{\partial x} \xrightarrow{\mathcal{F}} (2\pi i \xi)\, \hat{u}(\xi,t)$$

$$= 2\pi i \xi \cdot \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t}.$$

$$(-2\pi i x)\cdot \frac{\partial u(x,t)}{\partial x} \xrightarrow{\mathcal{F}} \frac{d\left(2\pi i \xi \hat{f}(\xi)\, e^{-4\pi^2\xi^2 t}\right)}{d\xi}$$

By Inversion formula,

$$-2\pi i x \cdot \frac{\partial u(x,t)}{\partial x} = \int_{\mathbb{R}} \frac{d\left(2\pi i \xi \hat{f}(\xi)\, e^{-4\pi^2\xi^2 t}\right)}{d\xi}\, e^{2\pi i \xi x}\, d\xi$$

So

$$\sup_{\substack{x\in\mathbb{R}\\ 0<t<T}} 2\pi \left| x \cdot \frac{\partial u}{\partial x}(x,t) \right| \leq \sup_{\substack{x\in\mathbb{R}\\ 0<t<T}} \int_{\mathbb{R}} \underbrace{\left| \frac{d\left(2\pi i \xi \hat{f}(\xi)\, e^{-4\pi^2\xi^2 t}\right)}{d\xi} \right|}_{(***)} d\xi$$

$$= \sup_{0<t<T} \int_{\mathbb{R}} \left| (***) \right| d\xi$$

Notice that

$$(***) = 2\pi i \left[\left(\xi \hat{f}(\xi)\right)' e^{-4\pi^2\xi^2 t} + \xi \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} \cdot (-8\pi\xi t) \right]$$

Hence

$$\left| (***) \right| \leq 2\pi \cdot \left| \left(\xi \hat{f}(\xi)\right)' \right| + 16\pi^2 \left| \xi^2 \hat{f}(\xi) \right| \cdot t .$$

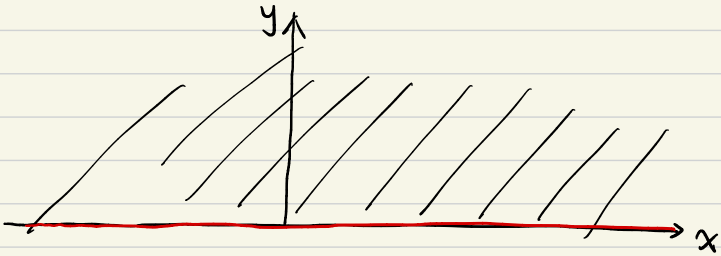
Hence

$$\sup_{0<t<T} \int_{\mathbb{R}} \left| (***) \right| d\xi$$

$$\leq 2\pi \int \left| \left(\xi \hat{f}(\xi)\right)' \right| d\xi + 16\pi^2 \int_{\mathbb{R}} \left| \xi^2 \hat{f}(\xi) \right| d\xi \cdot T$$

$$< \infty .$$

Application 2: Steady state heat equation on the upper half plane.



$u = u(x, y)$ temperation distribution

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0. & (1) \\ u(x, 0) = f(x). & (2) \end{cases}$$

Now we use Fourier transform to derive a solution by some formal arguments.

Taking Fourier transform in x-variable in ① gives

$$(2\pi i \xi)^2 \hat{u}(\xi, y) + \frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} = 0$$

That is,
$$\frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} - 4\pi^2 \xi^2 \hat{u}(\xi, y) = 0$$

For fixed ξ, the above is a linear 2nd order ODE.

The general solution is

$$\hat{u}(\xi, y) = A(\xi)\, e^{-2\pi|\xi|y} + B(\xi)\, e^{2\pi|\xi|\cdot y}$$

We remove the second part since it is rapidly increasing.

Therefore

$$\hat{u}(\xi, y) = A(\xi)\, e^{-2\pi|\xi|y}.$$

Taking $y=0$,

$$\hat{u}(\xi, 0) = A(\xi) = \hat{f}(\xi)$$

That is

$$\hat{u}(\xi, y) = \hat{f}(\xi) \cdot e^{-2\pi|\xi|y}.$$

Now we introduce the Poisson kernel on the upper half plane:

$$P_y(x) = \frac{1}{\pi} \cdot \frac{y}{x^2+y^2}, \qquad x \in \mathbb{R}, \quad y > 0$$

Claim: $\widehat{P_y}(\xi) = e^{-2\pi|\xi|y}$ for $y>0$.

Hence $\hat{u}(\xi, y) = \hat{f}(\xi) \cdot \widehat{P_y}(\xi)$

$= \widehat{f * P_y}(\xi)$.

By Inversion formula,

$u(x,y) = f * P_y(x), \quad x \in \mathbb{R}, \quad y>0.$

Lem 1.

(1) $\displaystyle\int_{-\infty}^{\infty} e^{-2\pi|x|y} e^{-2\pi i \xi x} dx$

$= P_y(\xi)$

(2) $\displaystyle\int_{-\infty}^{\infty} P_y(\xi) e^{-2\pi i \xi x} d\xi = e^{-2\pi|x|y}$

Pf. Remember for $a>0$,

$$e^{-a|x|} \xrightarrow{\mathcal{F}} \frac{2a}{a^2+4\pi^2\xi^2}$$

Letting $a = 2\pi y$ gives

$$e^{-2\pi y|x|} \xrightarrow{\mathcal{F}} \frac{2\cdot 2\pi y}{(2\pi y)^2+4\pi^2\xi^2} = \frac{1}{\pi}\cdot\frac{y}{y^2+\xi^2}$$
$$= \mathcal{P}_y(\xi).$$

This proves ①.

By ① and Inversion formula,

$$\int \mathcal{P}_y(\xi)\, e^{+2\pi i \xi x}\, d\xi = e^{-2\pi y|x|}$$

Taking complex conjugate on both sides gives ②